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Temperature Profiles for Diffusion Problem Precise Solutions Using Heat Balance Integral Method

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In the present paper, the heat balance integral method is applied for the solution of semi-infinite solid heat conduction problems subjected to prescribed polynomial type time-variable temperature or heat flux boundary conditions. A real *n* value exponent temperature profile is used. The obtained results are compared with literature classical solutions. For the first time in the literature, an algebraic expression for the *n* exponent, which depends on the boundary conditions applied to the problem, is proposed. Although the heat balance integral method is much easier to implement, the precision obtained using the proposed temperature profile is of the same order of magnitude of the classical literature solutions. The temperature distributions obtained from a numerical benchmark test case are used in the comparison with the present solutions. This study also discusses the restrictions of the real *n* exponent temperature profile heat balance integral method solutions.

Nomenclature

A = number of recurrence of repeated integral of error function
B = argument of repeated integral of error function
E = nondimensional global error
F = prescribed heat flux problem
k = thermal conductivity, W/m K
L = characteristic length, m
m_F = nondimensional prescribed heat flux exponent
m_T = nondimensional prescribed temperature exponent
n = temperature profile exponent
q = dimensional prescribed heat flux, W/m^2
Q_o = nondimensional prescribed heat flux coefficient
Q_P = prescribed time-variable nondimensional heat flux
Q_S = nondimensional surface heat flux
R = dummy variable
T = temperature, K
t = time, s
T_o = initial temperature, K
T_P = dimensional prescribed temperature, K
T_R = reference temperature, K
X = dimensional length, m
x = nondimensional length
Z = dummy variable
α = heat diffusivity, m^2/s
β = nondimensional normalized error
δ = nondimensional heat penetration depth
θ = nondimensional temperature
θ_o = nondimensional prescribed temperature coefficient
θ_P = nondimensional time-variable prescribed temperature
θ_S = nondimensional surface temperature
τ = nondimensional Fourier time
φ = time-dependent temperature profile parameter

Introduction

CONDUCTION heat transfer is a very important phenomenon to engineering science. Since Fourier's work [1], many mathematical methods have been developed to help to understand and predict the thermal behavior of different materials. Some of these mathematical methods are considered classic and can be easily obtained from several textbooks, such as Carslaw and Jaeger [1] and Ozisik [2]. Some of the classic methods are considered exact, such as the separation of variables and Fourier and Laplace transforms, while others, like variational and integral methods, are considered approximate. Kiwan et al. [3] and Saleh and Al-Nimr [4] are among the researchers who recently developed new solution techniques based on the classical methods.

The heat balance integral method (HBIM) is a very useful approximate analytical method for heat conduction problems. The HBIM was developed by Goodman, based on the Karman–Pohlhausen method and, as expressed by Goodman [5], "...although approximate, [the method] provides accuracy adequate for engineering purpose and has the distinct advantage of reducing the problem from one requiring the solution of a partial differential equation, which is relatively difficult, to one requiring the solution of an ordinary differential equation, which is relatively easy". The accuracy of the HBIM solution is directly related to the choice of a basic temperature profile that is used. Historically, polynomial functions are selected, but as observed by Goodman [6], "... there is not a priori guarantee that increasing the order of the polynomial will improve the accuracy".

As registered by Wood et al. [7], since the Goodman original work [6], several papers have been published concerning the improvement of the method accuracy. Two major approaches are usually employed in those papers: modification of the temperature profile used on the HBIM, in an attempt to minimize the approximation errors, or spatial subdivision of the interest domain, with the use of low-order piecewise temperature profiles. Sadoun et al. [8] justify the use of the piecewise approximation as the following: "... despite a large number of investigations, there is, unfortunately, no systematic procedure to choose the most appropriate profile [to be used at the HBIM]... then, works were oriented towards a decrease on the accuracy dependence of the method on arbitrary profile". This approach usually leads to algorithms very similar to the classical numerical ones, such as finite difference and finite volume.

Braga et al. [9], in 2003, proposed a temperature profile for a semi-infinite conduction heat transfer problem with an ablation boundary condition, where the exponent of the proposed function was not an integer number. The use of this profile resulted in better comparison with exact solutions than the classical polynomial functions (integer exponent) approach. In 2004, Braga et al. [10] continued the research with the noninteger exponent profile for the finite ablative problem. Based on these results, Braga et al. [11,12] presented in 2005 two papers concerning the new approach for the HBIM for a semi-infinite solid subjected to several boundary conditions: prescribed heat flux,

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prescribed temperature (Braga et al. [11]), and convection (Braga et al. [12]). All of them presented very good results. Later, in 2009, Braga and Mantelli [13] applied the same approach for finite heat conduction problems subjected to prescribed heat flux and temperature, again presenting good results. Following this trend, Meyers [14] and Hristov [15], based on Braga et al.'s [11] work, used different methods to determine ideal values of the exponent of the temperature profile function.

In the present paper, n exponent temperature profiles are also used to solve the one-dimensional heat conduction semi-infinite solid problem subjected to polynomial time-variable prescribed temperature or flux. For the first time in the literature, by means of the comparison of the present HBIM with the classical exact Laplace solutions [1], an expression for the determination of the appropriated n value based on the boundary condition considered, which minimizes the difference between HBIM and classical results, is obtained. This expression is tested using a numerical test case. As demonstrated in this paper, the precision of the HBIM can be improved substantially by the selection of the n exponent temperature profile; therefore, this method, which is simple to implement and requires little computational effort, can be applied to several problems, such as inverse heat transfer, where analytical approximate methods were not considered appropriate.

Physical Modeling

A one-dimensional semi-infinite solid, made of an isotropic material with constant properties, is considered in this work. The body is assumed to be at a uniform temperature until heat is applied in the left (free) surface. Two different heating conditions are considered. One is a time-variable heat flux Q_P and the other is a time-variable temperature θ_P. One should note that these boundary conditions are quite unusual for classical methods, which are usually based on time constant conditions. Another important aspect to observe is that they can also be considered as limits of a convection heat transfer case. At the initial time, when convection is applied to the free surface, the temperature difference between the surface and the environment is high, so that the convection problem tends to the one with a prescribed heat flux boundary condition. In contrast, as time goes to infinity, the temperatures of the body and of the surface tend to steady-state conditions, which can be represented by a prescribed temperature condition. So the analysis performed in the present paper can be extended to convection boundary condition problems.

The applied heat is conducted through the material generating two different sections: a heated region, in which the temperature is affected by the heating imposed at the free surface of the material, and the virgin region, where the material remains at the initial temperature. The distance between the left surface of the material and the front end of these regions is named the heat penetration depth δ. Figure 1 illustrates the physical model scheme adopted for the prescribed heat flux case. One should note that no heat source or sink is considered in this problem, although the introduction of a uniformly spread heat source or sink would only increase or decrease the temperature level of the solution, not affecting its distribution.

Mathematical Modeling

In this section, the mathematical model used to predict the thermal behavior of the problem considered is developed.

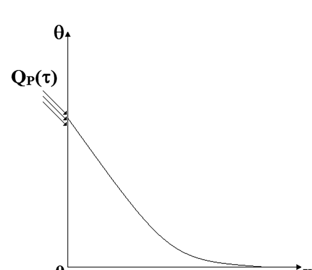

Fig. 1 Physical model: prescribed heat flux case.

A heat balance over the body (see Fig. 1) leads to the following well-known transient nondimensional differential heat equation [16], where $\theta = (T - T_o)/(T_R - T_o)$, T_o is the initial temperature (in Kelvins), and T_R is any arbitrary reference temperature (in Kelvins):

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^2} \tag{1}$$

In this equation, τ is the Fourier nondimensional time, defined as $\tau = \alpha t/L^2$, where α is the heat diffusivity (in meters squared per second), t is the time (in seconds), and L corresponds to a characteristic length (in meters). Also, x is defined as $x = X/L$, where X is a dimensional length (in meters). In actual problems, L represents the length of the solid within which the heat transfer problem can be considered semi-infinite. Two different situations can be considered for the right side of the solid. First, the solid is considered semi-infinite and, in the infinite limit, the temperature reaches the initial condition T_o, or

$$\lim_{x\to\infty} \theta = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\partial \theta}{\partial x} = 0 \tag{2}$$

Second, there are two distinct regions: one which is affected by the heat applied in the free surface, varying from $x = 0$ to $x = \delta$, and the other is not affected by the heat, denominated as the virgin region, which varies from $x = \delta$ to $x = \infty$. These conditions can be modeled as

$$\theta|_{x=\delta} = 0 \quad \text{and} \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=\delta} = 0 \tag{3}$$

In the left free surface of the material, two boundary conditions are considered, depending on the heating method used. For the prescribed heat flux case, the following equation is used:

$$-\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = Q_P \tag{4}$$

where Q_P is the prescribed nondimensional heat flux, defined as $Q_P = qL/[k(T_R - T_o)]$, where k is the thermal conductivity (in watts per meter times Kelvin) and q is the dimensional prescribed heat flux (in watts per meter). In the present work, the heat flux is considered time dependent, as it will be treated later.

On the other hand, for the prescribed temperature heating method, the expression is

$$\theta|_{x=0} = \theta_P \tag{5}$$

where θ_P is the prescribed nondimensional temperature, defined as $\theta_P = (T_P - T_o)/(T_R - T_o)$, with T_P as the dimensional prescribed temperature (in Kelvins). As for the heat flux case, time-variable expressions are considered for θ_P, as will be presented later in this paper.

Solutions

In this section, two different analytical methods will be used to solve the problem under investigation.

Classical Solution

The Laplace transform technique, as presented in the literature by many of the traditional conduction heat transfer books ([1,16], for instance), is used as the classical solution. This solution is applied for the two boundary condition cases under investigation.

Time-Variable Prescribed Heat Flux Boundary Condition

The free surface prescribed heat flux has the form $Q_o \tau^{m_F}$, which can express several time-variable boundary conditions, depending on the parameter m_F, including the constant heat flux for $m_F = 0$. Solving Eq. (1), subjected to the boundary conditions given by Eqs. (2) and (4), using the Laplace transform technique, the following nondimensional expression is obtained (see [17] for more details):

$$\theta = Q_o \Gamma(m_F + 1)(4\tau)^{m_F+1/2} \text{erfc}\left(2m_F + 1, \frac{x}{\sqrt{4\tau}}\right) \tag{6}$$

where the erfc(a, b) is the repeated integral of the error function that, according to the literature [18], takes the form

$$i^a \text{erfc}(b) = \int_b^\infty i^{a-1} \text{erfc}(z)\, \mathrm{d}z \tag{7}$$

In this expression, the parameter a represents the number of integrations performed (number of recurrences). If $a = 1$, the preceding expression has the same form of the erfc(b). If $a = 2$, the preceding expression is integrated once, while if $a = 3$, it is integrated two times, etc. As the function erfc(a, b) is defined only for integer a numbers, the parameter $2m_F + 1$ must present integer values. This means that m_F, in Eq. (6), must be selected so that this condition is satisfied. In this same equation,

$$\Gamma(r) = \int_0^\infty z^{r-1} e^{-z}\, \mathrm{d}z$$

is the well-known Gamma function [18], where z and r are dummy variables. One can see that it is hard to implement this classical solution for this kind of problem due to these complicated functions.

Time-Variable Prescribed Temperature Boundary Condition

Similarly, the free surface time-variable prescribed temperature has the form $\theta_o \tau^{m_T}$, which can express several time-variable boundary conditions, depending on the parameter m_T. Solving the problem given by the partial differential equation [Eq. (1)], subjected to the boundary conditions given by Eqs. (2) and (5), using the Laplace transform technique, and after considerable manipulation (see [17] for details), the following nondimensional solution is obtained:

$$\theta = \theta_o \Gamma(m_T + 1)(4\tau)^{m_T} \text{erfc}\left(2m_T, \frac{x}{\sqrt{4\tau}}\right) \tag{8}$$

where, as for Eq. (6), $2m_T$ must present an integer value.

Heat Balance Integral Method Solution

To apply the HBIM, the integral form of the heat conduction differential equation is obtained by the integration of Eq. (1) with respect to the variable x (position) from the free solid surface ($x = 0$) up to the heat penetration depth ($x = \delta$). By doing so, the following equation is obtained:

$$\int_0^\delta \frac{\partial \theta}{\partial \tau}\, \mathrm{d}x = \left.\frac{\partial \theta}{\partial x}\right|_{x=\delta} - \left.\frac{\partial \theta}{\partial x}\right|_{x=0} \tag{9}$$

As δ is a time-dependent variable, the Leibniz rule is used and Eq. (9) is rearranged as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\delta \theta\, \mathrm{d}x - \theta|_{x=\delta} \frac{\partial \delta}{\partial \tau} = \left.\frac{\partial \theta}{\partial x}\right|_{x=\delta} - \left.\frac{\partial \theta}{\partial x}\right|_{x=0} \tag{10}$$

At this point, an appropriate function has to be selected for the temperature distribution profile in the solid. This function must have a good agreement with the expected temperature distribution, with the space boundary conditions, and must present time-dependent parameters, which will be determined using the remaining boundary conditions. In this work, the following profile is considered for the temperature distribution:

$$\theta = \varphi \left(1 - \frac{x}{\delta}\right)^n \tag{11}$$

where φ is a time-variable parameter calculated through the free surface boundary condition. Physically, this parameter represents the surface temperature level of the material, while the exponent n establishes the shape of the temperature profile along the solid being arbitrarily selected. The best selection of the n value and its implications will be explained later in this paper. The profile represented by Eq. (11) naturally satisfies the boundary conditions given by Eq. (3). Its profile is valid only for the range $0 \le x \le \delta$ and is considered zero elsewhere. Substituting Eq. (11) in Eq. (10), one gets the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\varphi \delta}{n + 1}\right) = \frac{\varphi n}{\delta} \tag{12}$$

Heat Balance Integral Method Prescribed Heat Flux Problem

The HBIM prescribed heat flux problem is discussed in this section. The temperature distribution [Eq. (11)] is substituted in the heat flux boundary condition [Eq. (4)], obtaining the following equation:

$$\frac{\varphi_F n_F}{\delta_F} = Q_P \tag{13}$$

where the F subindex indicates the heat flux problem.

Solving for φ_F and substituting this parameter in Eq. (12), the following differential equation is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{Q_P \delta_F^2}{n_F(n_F + 1)}\right) = Q_P \tag{14}$$

This equation can be easily solved for the heat penetration depth, resulting in

$$\delta_F = \sqrt{\frac{n_F(n_F + 1)}{Q_P} \int_0^\tau Q_P\, \mathrm{d}\bar{\tau}} \tag{15}$$

Considering $Q_P = Q_o \tau^{m_F}$, substituting in the last equation and solving for φ_F, one gets

$$\varphi_F = Q_o \tau^{m_F+1/2} \sqrt{\frac{(n_F + 1)}{n_F(m_F + 1)}} \tag{16}$$

Making all the necessarily algebraic manipulation, one gets the temperature profile

$$\theta = Q_o \tau^{m_F+1/2} \sqrt{\frac{(n_F + 1)}{n_F(m_F + 1)}} \times \left(1 - \frac{x}{\sqrt{\{[n_F(n_F + 1)]/(m_F + 1)\}\tau}}\right)^{n_F} \tag{17}$$

One should note that the only unknown variable of this last equation is the n_F parameter.

Prescribed Temperature Problem

Similar to the procedure adopted in the last section, the prescribed temperature problem is solved by substituting the temperature distribution given by Eq. (11) in the prescribed temperature boundary condition expression, presented in Eq. (5), obtaining the following expression:

$$\varphi_T = \theta_P \tag{18}$$

where the T subindex indicates the prescribed temperature problem.

Substituting Eq. (18) in Eq. (12), the following differential equation is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\theta_P \delta_T}{n_T + 1}\right) = \frac{\theta_P n_T}{\delta_T} \tag{19}$$

This equation can be solved for the heat penetration depth, resulting in

$$\delta_T = \sqrt{\frac{2n_T(n_T+1)}{\theta_P^2}\int_0^\tau \theta_P^2\,\mathrm{d}\bar{\tau}} \tag{20}$$

As already mentioned, the prescribed temperature is considered time dependent, given by $\theta_P = \theta_o \tau^{m_T}$. Substituting in the last equation, solving for heat penetration depth, one gets

$$\delta_T = \sqrt{\frac{2n_T(n_T+1)}{2m_T+1}\tau} \tag{21}$$

Making the appropriate substitutions, the following expression is obtained for the temperature distribution:

$$\theta = \theta_o \tau^{m_T}\left(1 - \frac{x}{\sqrt{[2n_T(n_T+1)]/(2m_T+1)\tau}}\right)^{n_T} \tag{22}$$

As for the prescribed heat flux case, the only unknown variable that remains is the n_T parameter.

Obtaining the *n* Parameter

As described before, the n parameter is the exponent of the expression for the temperature distribution profile adopted and, usually, is arbitrarily selected. Historically [15], several authors have used integer values, such as 2, 3, or 4. In the present work, the n value is free to take any real value. To find out the best value for n, a comparison between the classical and the HBIM solutions is performed.

Prescribed Heat Flux

The free surface temperatures, obtained from the HBIM and the classical solutions by the substitution of $x = 0$ in Eqs. (6) and (17), are equated and solved for n_F, resulting in the following expression:

$$n_F = \frac{\Gamma(m_F+3/2)^2}{(m_F+1)\Gamma(m_F+1)^2 - \Gamma(m_F+3/2)^2} \tag{23}$$

Therefore, Eqs. (15) and (16) can be rewritten as, respectively,

$$\delta_F = \frac{\Gamma(m_F+1)\Gamma(m_F+3/2)}{(m_F+1)\Gamma(m_F+1)^2 - \Gamma(m_F+3/2)^2}\sqrt{\tau} \tag{24}$$

$$\varphi_F = Q_o \tau^{m_F+1/2}\frac{\Gamma(m_F+1)}{\Gamma(m_F+3/2)} \tag{25}$$

Figure 2 presents plots of the exponent n_F and of heat penetration depth divided by the square root of the Fourier time $(\delta_T/\tau^{1/2})$ as a function of the exponent of the prescribed heat flux m_F. From this figure, one can observe a linear behavior between n and m and an asymptotic trend of the modified heat penetration depth. These aspects show that there is no unique best n value for any heat flux condition (represented by m_F parameters) but a best n expression. Braga et al. [11] and Hristov [15] studied a temperature profile equivalent to $m_F = 0$, obtaining the parameters $n_F = 3.659$ and $\delta_F = 4.129\sqrt{\tau}$, which agree perfectly well with the present work results.

Prescribed Temperature

Similarly, for the prescribed temperature case, the solutions are compared. Equating the temperature distributions for the HBIM and classical solution and solving for n_T, the following expression is obtained:

$$n_T = \frac{2\Gamma(m_T+1)^2}{(2m_T+1)\Gamma(m_T+1/2)^2 - 2\Gamma(m_T+1)^2} \tag{26}$$

Substituting this equation in Eq. (21), one gets

$$\delta_T = \frac{2\Gamma(m_T+1/2)\Gamma(m_T+1)}{(2m_T+1)\Gamma(m_T+1/2)^2 - 2\Gamma(m_T+1)^2}\sqrt{\tau} \tag{27}$$

Figure 3 presents the temperature profile exponent n_T and the heat penetration depth divided by the square root of the Fourier time $(\delta_T/\tau^{1/2})$ as a function of prescribed temperature expression exponent m_T. The same observations made for the prescribed heat flux case can also be made for this case: n is linear with respect to m, and the modified heat penetration depth shows an asymptotic behavior. The numerical values found for $m_T = 0$ (i.e., $n_T = 1.751$ and $\delta_T = 3.105\sqrt{\tau}$) agree perfectly well with those reported in [11,15].

It is important to note that, for the HBIM solutions, the variables $2m_F$ and $2m_T$ can assume any positive real value, in contrast with the classical solutions where these variables should be integer values. This possibility leads to a more general use of the HBIM solution, including problems where the boundary conditions are time variable. HBIM can also be applied as direct solutions of inverse heat transfer problems, where the parameter to be obtained is a boundary condition.

Error Analysis

According to Langford's method [19], an error inherent to the use of any one-dimensional transient heat conduction approximated solution θ is obtained through a least-squares estimate of the original equation:

$$E = \int_0^\delta \left(\frac{\partial\theta}{\partial\tau} - \frac{\partial^2\theta}{\partial x^2}\right)^2 \mathrm{d}x \tag{28}$$

If θ is the exact solution, then E is identically zero. Squaring the terms inside the brackets in the preceding equation prevents algebraic cancellation of errors of opposite sign and magnifies local differences between the solution and the approximation, increasing the sensibility of the method to detect errors.

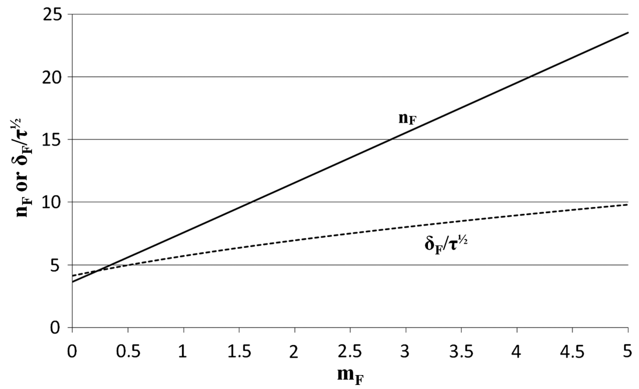

Fig. 2 Exponent profile and modified heat penetration depth at the prescribed heat flux case as a function of boundary exponent m_F.

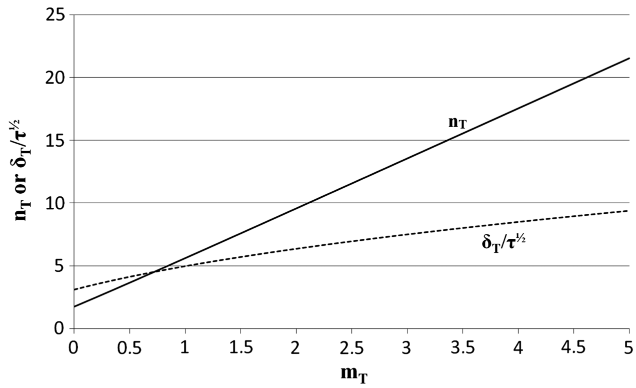

Fig. 3 Exponent profile and modified heat penetration depth for the prescribed temperature case as a function of exponent m_T.

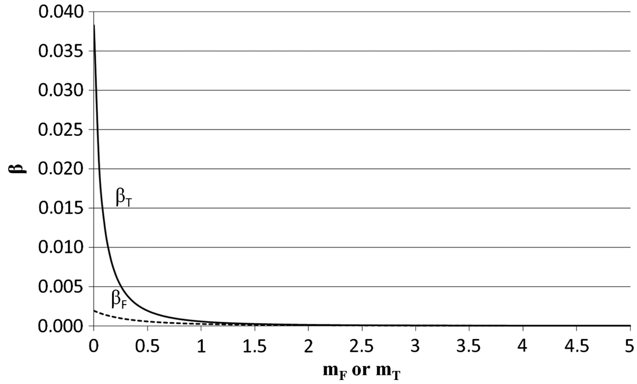

Fig. 4 Normalized error as a function of the boundary condition exponent.

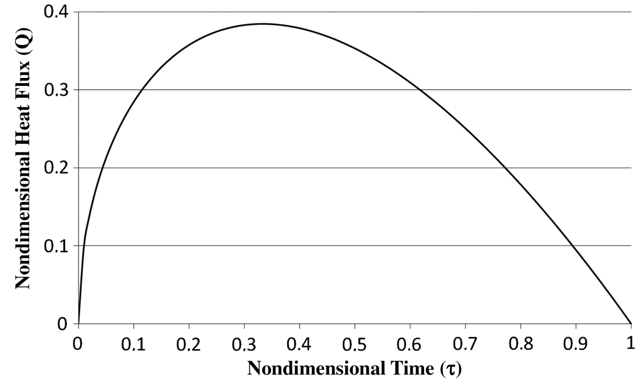

Fig. 5 Boundary condition heat flux applied to free solid surface.

After performing the appropriated derivatives of the HBIM solution [Eq. (11)] and substituting in Eq. (28), the space integral is performed. Braga [17] shows that the error can be expressed by

$$E = \frac{\theta_S^2 \beta}{\tau^{3/2}} \tag{29}$$

where θ_s is the surface temperature. Parameter β is nondimensional and depends only on the boundary conditions exponent (m_t and m_f for temperature and heat flux, respectively). This parameter can be understood as a normalized error parameter; β low values indicate more accurate solutions. By Eq. (29), one notes that increasing time decreases the total error E. Braga [17] presents explicit equations for β parameters for heat flux and temperature boundary conditions that are very long and complicated, and they will not be presented here. Instead, plots of β as a function of m are shown in Fig. 4.

One can observe in Fig. 4 that the increase of the m value represents a decrease of the β value and, consequently, of the global error of the solution. The β value is always lower than 0.04, which indicates a precise solution, compatible with most of the engineering application problems. The worst values are obtained at $m = 0$, which is associated with the constant time boundary condition.

Test Case

To compare the classical with the HBIM solutions, a test case solved numerically is presented. A heat flux problem with a boundary condition in the form of $Q_P = Q_1\tau^{m_1} + Q_2\tau^{m_2}$ was selected for this study. This form of the free surface heat flux can represent several physical situations according to the numerical values used for the Q and m parameters. Table 1 values are adopted for the present study, where the Q_1 and Q_2 parameters represent positive and negative unitary heat fluxes, respectively, while m_1 and m_2 are selected as $\frac{1}{2}$ and $\frac{3}{2}$, respectively, so that $2m_1$ and $2m_2$ are both integer values, as required by the classical solutions employed for comparison. Figure 5 shows a plot of the time distribution of the heat flux boundary condition adopted for this study. One can see that the heat flux starts from zero ($\tau = 0$) and increases, reaching a maximum value, and then it decreases slowly until it reaches the zero level again for the nondimensional time $\tau = 1$.

It is well known that the analytical solution of problems subjected to composed boundary conditions is obtained by splitting the problem in two [16]. For the present case, one problem corresponds to the boundary condition $Q_{P1} = Q_1\tau^{m_1}$ and the other to $Q_{P2} = Q_2\tau^{m_2}$. The final solution is obtained as the sum of the solutions of these two problems. This procedure is applied for the classical method and HBIM. For both cases, θ_1 corresponds to the solution of the first problem and θ_2 to the second problem, and the problem solution is the sum of these two solutions, i.e., $\theta = \theta_1 + \theta_2$.

Table 1 Numerical values for test case

Value	Parameter
Q_1	1
Q_2	-1
m_1	$1/2$
m_2	$3/2$

One should note that one of the limitations of the classical solutions is that they can only be applied for cases where $2m_1$ and $2m_2$ are both integer values, as already mentioned. Actually, this restriction is not applied to the HBIM solution.

For Table 1 parameters, the value of $n = 5.615$ is for the θ_1 solution and $n = 9.572$ for θ_2.

For these cases, the classical solutions are

$$\theta_1 = \Gamma\left(\frac{3}{2}\right)(4\tau)\mathrm{erfc}\left(2, \frac{x}{\sqrt{4\tau}}\right) \tag{30}$$

and

$$\theta_2 = -\Gamma\left(\frac{5}{2}\right)(4\tau)^2\mathrm{erfc}\left(4, \frac{x}{\sqrt{4\tau}}\right) \tag{31}$$

while the HBIM solutions are

$$\theta_1 = \begin{cases} \tau\frac{\Gamma(3/2)}{\Gamma(2)}\left(1 - \frac{[(3/2)\Gamma(3/2)^2 - \Gamma(2)^2]x}{\Gamma(3/2)\Gamma(2)\sqrt{\tau}}\right)^{5615}, & x \le \frac{\Gamma(3/2)\Gamma(2)\sqrt{\tau}}{[(3/2)\Gamma(3/2)^2 - \Gamma(2)^2]} \\ 0, & x > \frac{\Gamma(3/2)\Gamma(2)\sqrt{\tau}}{[(3/2)\Gamma(3/2)^2 - \Gamma(2)^2]} \end{cases} \tag{32}$$

and

$$\theta_2 = \begin{cases} -\tau^2\frac{\Gamma(5/2)}{\Gamma(3)}\left(1 - \frac{[(5/2)\Gamma(5/2)^2 - \Gamma(3)^2]x}{\Gamma(5/2)\Gamma(3)\sqrt{\tau}}\right)^{9572}, & x \le \frac{\Gamma(5/2)\Gamma(3)\sqrt{\tau}}{[(5/2)\Gamma(5/2)^2 - \Gamma(3)^2]} \\ 0, & x > \frac{\Gamma(5/2)\Gamma(3)\sqrt{\tau}}{[(5/2)\Gamma(5/2)^2 - \Gamma(3)^2]} \end{cases} \tag{33}$$

Figure 6 shows a plot of the nondimensional temperatures obtained for both the classical method and HBIM as a function of the nondimensional position for several nondimensional times τ varying from 0.01 to 1. In the position $x = 0$, one can see the temperature variation resulting from the variable heat flux applied to the free surface of the semi-infinite solid. The surface temperature increases to the level around $\theta \approx 0.3$ for $\tau \approx 0.7$, decreasing to $\theta \approx 0.25$ for $\tau = 1$. This figure also shows the heat penetration inside the solid. The curves obtained for both methods compare very well, being almost indistinguishable for small values of the x and τ parameters. For larger values of these parameters, although the comparison is still very good, the curves show differences. This happens because the parameter n is obtained by the expression resulting from equating the free surface temperature equations, and so it is adjusted to the surface ($x = 0$). Therefore, departing from the surface, as x tends to 1, the difference between the results increases.

Figure 7 presents plots for the nondimensional temperature θ as a function of the nondimensional time τ for four nondimensional

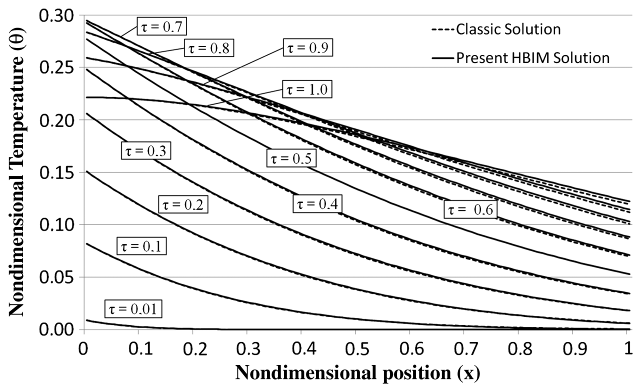

Fig. 6 Comparison of HBIM present and classical nondimensional temperature solutions as a function of the nondimensional position, for several nondimensional times.

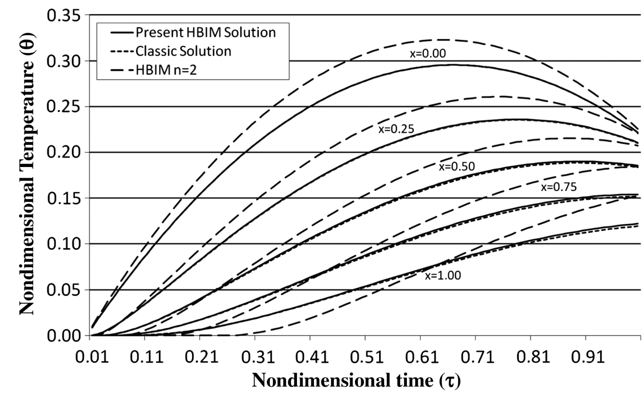

Fig. 9 Comparison among classical and HBIM nondimensional temperature solutions (present and literature, for $n = 2$) as a function of nondimensional time.

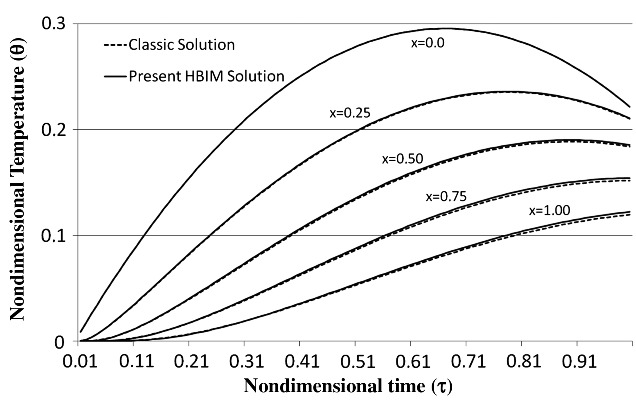

Fig. 7 Comparison of present HBIM and classical nondimensional temperature solutions as a function of the nondimensional time, for several nondimensional positions.

different positions ($x = 0$, 0.25, 0.5, 0.75, and 1). One can see from this plot that the larger temperature variation is observed in $x = 0$, decreasing as x increases. It is very clear that the larger difference between the classic and HBIM solutions is observed for $x = 1$, as already mentioned.

Figure 8 presents the plot of the difference of the nondimensional temperature $\Delta\theta$ between the HBIM and classical solutions as a function of the nondimensional time parameter τ. It can be observed that the difference is always less than 0.003. Although the difference between these methods increases with increasing time and increasing nondimensional length, the small difference observed shows that the

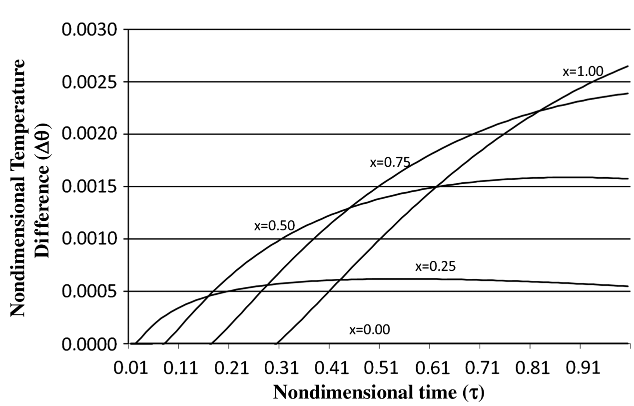

Fig. 8 Difference between HBIM and classical solution for nondimensional temperature as a function of nondimensional time for several nondimensional positions.

HBIM approximate solution, which is simpler and faster to implement, can be used with precision, substituting classical methods, especially when the classical solution cannot be applied.

The precision obtained with the HBIM using the Eq. (11) temperature profile, in which the n exponent is given by Eqs. (23) and (26) (depending on the applied boundary conditions) as proposed in the present paper, is much larger than that observed in the literature. Goodman [6], Ozisik [2], and Histrov [15], among others, used polynomial expressions with exponent 2 for the temperature profile when applying HBIM. Figure 9 presents a comparison of results obtained for the HBIM using the expression proposed in the present paper and the literature second-degree polynomial with the classical solutions. In this plot of the nondimensional temperature as a function of the nondimensional time, the temperature distributions for several nondimensional positions of the solid are presented. One can observe that, while the present proposed solution is very similar to the classical (exact) solution, being indistinguishable for $x = 0$ and $x = 0.25$, the literature and classical solution can be very different from each other, especially for $x = 1$ and $\tau = 1$.

One should note that the HBIM as presented in the literature predicts the temperature distribution within a solid subjected to a very fast heating condition (very large m value) very poorly, while the n exponent temperature profile approach presented in this paper shows results as precise as those presented for this work test case.

Conclusions

In the present work, the HBIM method is applied for the solution of the semi-infinite heat conduction problem, subject to variable heat flux or temperature boundary conditions in its free surface. Real number n exponent temperature profiles are employed. Analytical expressions, depending on the boundary conditions applied to the free surface, are developed to determine the best n parameter value. This expression fulfills the lack, in the literature, of a systematic procedure to be used to select the temperature profiles, which minimizes the error, when compared with exact solutions. It was shown that the correct selection of the n parameter reduces the HBIM solution errors to a minimum level. It is also important to note that the classical solutions are not able to deal with noninteger polynomial boundary conditions. On the other side, HBIM can provide analytical solutions, whatever the exponent of the polynomial boundary condition.

HBIM is well known to provide easy-to-implement solutions, in which precision is considerably worse than the classical methods. The present work shows that, using the n temperature profile obtained from the mathematical expressions developed, it is possible to apply HBIM and obtain solutions with the same order of precision of the classical methods. Therefore, the precision obtained in this work permits the use of the HBIM technique for many engineering problems, such as inverse heat transfer codes.

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